RG Topology
Institute for Algebra and Geometry
Winter term 2018

Current developments around the Atiyah conjecture

Workshop 11/19/2018 - 11/21/2018

Monday, 11/19/2018

Time	**Speaker**	**Title**
09:30–10:30	Roman Sauer	*Lück's approximation theorem*
10:30–11:00	Coffee break	
11:00–12:00	Holger Kammeyer	*The determinant and the approximation conjecture*
12:00–13:00	Lunch break	
13:00–14:00	Free discussion	
14:00–15:00	Sabine Braun	*The Atiyah conjecture*
15:00–15:30	Coffee break	
15:30–16:30	Michael Schrödl	*Sofic groups and approximation I*
16:30–17:30	Joint discussion on the background of the Atiyah conjecture	

Tuesday, 11/20/2018

Time	**Speaker**	**Title**
09:30–10:30	Steffen Kionke	*Sofic groups and approximation II*
10:30–11:00	Coffee break	
11:00–12:00	Hartwig Senska	*Von Neumann regular, $*$-regular rings and the $*$-regular closure*
12:00–13:00	Lunch break	
13:00–14:00	free discussion	
14:00–15:00	Alessandro Carderi	*Sylvester rank functions*
15:00–15:30	Coffee break	
15:30–16:30	Diego López Álvarez	*Epic $*$-regular R-rings and Sylvester rank functions*
16:30–17:30	Joint discussion on latest proof methods in the field	

Wednesday, 11/21/2018

Time	**Speaker**	**Title**
09:30–10:30	Benjamin Waßermann	*The centralizer dimension*
10:30–11:00	Coffee break	
11:00–12:00	Fabian Henneke	*The natural extensions and Sofic Lück approximation over arbitrary K with $\operatorname{char} K = 0$, part I*
12:00–13:00	Lunch break	
13:00–14:00	free discussion	
14:00–15:00	Benjamin Waßermann	*Sofic Lück approximation over arbitrary K with $\operatorname{char} K = 0$, part II and applications*
15:00–15:30	Coffee break	
15:30–16:30	Diego López Álvarez	*Around the Atiyah conjecture for one relator groups*
16:30–17:30	Joint discussion of possible new approaches to the problem	

Suggested outline of talks

Lück's approximation theorem. Explain the statement of Lück's approximation theorem for free finite type G-CW complexes with a residually finite group G. Explain why it has an equivalent formulation in terms of matrices $A \in M(k, l; \mathbb{Q}G)$ over the rational group ring $\mathbb{Q}G$. Prove the theorem, highlighting the two main ingredients *weak convergence* of spectral measures and Lück's *logarithmic bound* as a consequence of considering matrices with *integral* coefficients.

References: [7, Sections 4.1–4.3; 8; 9; 10, Chapter 13]

The determinant and the approximation conjecture. State the approximation conjecture. Explain how the need for a logarithmic bound as in the preceding talk leads naturally to the determinant conjecture. Show that a group G satisfies the approximation conjecture with respect to a cofinal system (G_i) if each quotient grup G/G_i satisfies the determinant conjecture. Prove the determinant conjecture for residually finite groups.

References: [7, Section 4.5; 10, Chapter 13]

The Atiyah conjecture. Introduce the (strong) Atiyah conjecture with coefficients in $\mathbb{Q} \subset K \subset \mathbb{C}$ and prove that it implies the Kaplansky zero divisor conjecture for the group ring KG if G is torsion-free. Show that if G satisfies the approximation conjecture with respect to (G_i) and each G/G_i is torsion-free and satisfies the Atiyah conjecture, then also G satisfies the Atiyah conjecture. Explain how this result broadens Linnell's class of groups the Atiyah conjecture is known for in case $K = \mathbb{Q}$ and report how the coefficients can be improved to $K = \overline{\mathbb{Q}}$.

References: [7, Sections 2.6 and 4.5; 10, Chapters 10 and 13]

Sofic groups and approximation I and II. Define sofic groups, give the sofic version of the approximation conjecture and indicate that for $R = \mathbb{Q}$ and residually finite G with a sofic approximation coming from a residual chain (G_i), Lück's classical proof goes through. Give an overview of Elek–Szabó's proof of the determinant conjecture for sofic groups, implying the sofic approximation conjecture with $R = \mathbb{Q}$. Explain how similarly to the preceding talk, the coefficients can be improved to $R = \overline{\mathbb{Q}}$. Show that for $R = \mathbb{C}$, Fuglede–Kadison determinants of the G/G_i-reductions of a fixed matrix can come arbitrarily close to zero. Conclude that hence "determinant methods" break down for $R = \mathbb{C}$. Say that in this workshop, we want to understand Jaikin–Zapirain's new algebraic approach to extend both the positive results on the (sofic) approximation conjecture and the Atiyah conjecture from $R = \overline{\mathbb{Q}}$ to $R = \mathbb{C}$.

References: [2; 3; 5, Sections 10.1–10.4]

Von Neumann regular, $*$-regular rings and the $*$-regular closure. Define von Neumann regular rings and sketch the proof of Proposition 3.1 in [6], which can be found in [4]. Define proper $*$-rings and show that, in such a ring, any von Neumann regular element admits a canonical *left-* and *right-projection*, as well as a canonical *relative inverse* (Proposition 3.2 in [6]). Define $*$-regular rings and prove that for any $*$-regular ring $\mathcal{U}$, any $*$-closed subring R admits a $*$-regular closure $\mathcal{R}(\mathcal{U}, R)$ (Proposition 6.1 in [1]). Compare $*$-regular closures with division closures.

References: [1; 4; 6, Sections 3.1–3.4]

Sylvester rank functions. Introduce the notion of a Sylvester matrix rank function over an algebra R, prove Proposition 5.1, and define the contravariant functor $\mathbb{P}$, assigning to each algebra R the space $\mathbb{P}(R)$ of rank functions over R. Explain what it means for a rank function to be *exact* and/or *faithful*. Define Sylvester module rank functions and explain their relation to matrix rank functions. Introduce the (pseudo)-metric induced by a rank function rk on R and the associated metric completion $\overline{R_{\mathrm{rk}}}$. Define regular Sylvester matrix rank functions, state Proposition 5.7 and sketch a proof of Proposition 5.8. Explain the ultrafilter construction as detailed in Section 5.7. State and sketch a proof of Proposition 5.11.

References: [6, Section 5]

Epic $*$-regular R-rings and Sylvester rank functions. Recall the definition of an epic ring homomorphism $f : S \to R$. State the equivalent characterization of epicity as given in Proposition 4.1. Introduce the dominion of a general morphism $f : R \to S$ and prove Corollary 4.3. Prove that the embedding of any $*$-subring into its $*$-regular closure (inside any ambient $*$-regular ring) is epic (Proposition 6.1) and deduce Corollary 6.2. Define epic $*$-regular R-rings and explain what it means for two such rings to be isomorphic. For a fixed $*$-ring R, introduce the subset $\mathbb{P}_{*reg}(R) \subseteq \mathbb{P}(R)$ of

$*$-regular rank functions and establish the natural $1:1$-correspondence between elements of $\mathbb{P}_{*reg}(R)$ and isomorphism classes of epic $*$-regular R-rings (Theorem 6.3). Explain the concept of general approximation of a countable group G and use Theorem 6.3 to derive the useful reformulation of Lück approximation (Theorem 6.6). Use the established techniques to conclude Proposition 9.5.

References: [6, Sections 4,6 and 9.2]

The Centralizer Dimension. Outline the strategy for the inductive proof of the Lück approximation conjecture (Section 2.2).
For a finitely generated group H, explain the H-module structure of the Hilbert space with basis given by a set X on which H acts freely and cofinitely, and elaborate on the natural $H \times H$-module structure on the corresponding space of Hilbert-Schmidt operators. Show that the Hilbert-Schmidt centralizer of an unbounded operator over $l^2(X)$ has a well-defined dimension. Sketch a proof of the limit formula for the centralizer dimension in the case where $X = G$ and $\{X_i\}_{i\in\mathbb{N}}$ is a sofic approximation of G, as given in Proposition 9.6. Also, sketch a proof of the inequality in Proposition 9.9. Use Proposition 9.9 and 9.6 to conclude the *Centralizer Dimension Formula* (Theorem 9.12). Highlight the subtle contribution of the *Small Eigenvalue Property.* Conclude the *Algebraic Eigenvalue Property* (Corollary 9.13).

References: [6, Sections 2.2 and 9]

The natural extensions and Sofic Lück approximation over arbitrary K with $\operatorname{char} K = 0$, part I. For an algebra R over a field K and a rank function rk on R, define its natural transcendental extension $\widetilde{\mathrm{rk}}$ on $R \otimes_K K(t)$ and list some of its properties (Proposition 7.5). State Proposition 7.7 without proof but explain how Corollary 7.8 follows from it. Similarly, define the natural algebraic extension $\widetilde{\mathrm{rk}}$ of rk on $R \otimes_K F$ if E/K is an algebraic field extension, show (sketch) Proposition 7.12 and explain how Corollary 7.13 follows from it. State Theorem 10.1, outline the idea of its inductive proof (remark that the base of the induction is the classic Approximation Theorem) and give sufficient details on the inductive step for algebraic extensions (Section 10.1).

References: [6, Sections 7.2,7.3,7.5 and 10.1]

Sofic Lück approximation over arbitrary K with $\operatorname{char} K = 0$, part II, and applications. Finish up the Proof of Theorem 10.1 by doing the inductive step for transcendental extensions (Section 10.2). Show that Theorem 1.3 is a corollary of 10.1. Finally, prove Theorem 1.1 and give further applications of Theorem 1.3, as outlined in Section 10.4, like the Strong Algebraic Eigenvalue Conjecture, the Center Conjecture, the Independence Conjecture and/or the Kaplansky Conjecture for sofic groups (over appropriate fields K).

References: [6, Sections 10.2-10.4 and 1]

References

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[2] J. Dodziuk, P. Linnell, V. Mathai, T. Schick, and S. Yates, *Approximating L^2-invariants and the Atiyah conjecture*, Comm. Pure Appl. Math. **56** (2003), no. 7, 839–873. Dedicated to the memory of Jürgen K. Moser. MR **1990479**

[3] Gábor Elek and Endre Szabó, *Hyperlinearity, essentially free actions and L^2-invariants. The sofic property*, Math. Ann. **332** (2005), no. 2, 421–441. MR **2178069**

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[5] A. Jaikin–Zapirain, *ℓ^2-Betti numbers and their analogues in positive characteristic*, Proceedings of St Andrews, to appear (2017). Available for download at `http://verso.mat.uam.es/~andrei.jaikin/preprints/surveyl2.pdf`.

[6] ———, *The Base Change In The Atiyah And The Lück Approximation Conjecture*, Preprint (2018). Available for download at `http://matematicas.uam.es/~andrei.jaikin/preprints/sac.pdf`.

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[8] S. Kionke, *The growth of Betti numbers and approximation theorems*, e-print (2017). arXiv:**1709.00769**

[9] W. Lück, *Approximating L^2-invariants by their finite-dimensional analogues*, Geom. Funct. Anal. **4** (1994), no. 4, 455–481, DOI 10.1007/BF01896404. MR **1280122**

[10] ———, *L^2-invariants: theory and applications to geometry and K-theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 44, Springer-Verlag, Berlin, 2002. MR **1926649**