Complex Hyperbolic Geometry

RTG Seminar Winter 2023

Fernando Camacho Cadena

Overview

Complex hyperbolic geometry is an analogue of (real) hyperbolic geometry and exhibits relations with many areas in algebra and geometry. The space is defined as the projectivization of the level set of a signature $(n, 1)$ Hermitian form on $\mathbb{C}^{n,1}$ and can be studied through Riemannian geometry, $\mathrm{CAT}(k)$ spaces, locally symmetric spaces, complex, Kähler, Heisenberg, contact and CR geometry, algebraic groups, among others.

The goal of the seminar is to gain an understanding of complex hyperbolic geometry, and then delve into some of its more recent developments regarding the construction of lattices in $\mathsf{PU}(n, 1)$ first through arithmetics, and then through triangle groups.

The first part of the seminar will consist in getting familiar with the definitions, the group of isometries, and the boundary. Then we will study interesting objects in complex hyperbolic space, namely totally geodesic submanifolds and bisectors. This will be followed by the symplectic and Kähler structure on the space, and the CR and contact structure on the boundary. For the end of the seminar, we will discuss more recent topics, namely arithmetic lattices, and triangle groups in $\mathsf{PU}(2, 1)$.

List of talks

The complete list of topics is below. The topics with an asterisk are more involved, and not all of them have to be covered (but we should cover some of them!). I explain some things that should be covered in each talk, but feel free to add or take away some parts as you see fit! Just be careful that you do mention the necessary material for the next talk.

1. **Definitions and models for $\mathbb{H}^n_{\mathbb{C}}$ (Maurice):** Give the definition of complex hyperbolic space, the metric, geodesics, angles, segments, its different models, boundary, and briefly describe its (variable) curvature. In the reference, only consider the $\mathbb{K} = \mathbb{C}$ case.

References: [BH10, Chapter II.10, Sections: *Real, Complex and Quaternionic Hyperboic n-spaces, The Curvature of* $\mathbb{K}H^n$*, The Boundary at Infinity and Horospheres in* $\mathbb{K}H^n$ (only *10.22*), *Other Models for* $\mathbb{K}H^n$].

2. **The group of isometries (Dani):** Show that the group of isometries of complex hyperbolic space is $\mathsf{PU}(n,1)$ (which coincides with the group of biholomorphisms of $\mathbb{H}^n_{\mathbb{C}}$), as well as complex conjugation, disscuss the transitivity of the action by $\mathsf{PU}(n,1)$, the stabilizer of a base point, hyperbolic isometries, the action on the boundary, classify the types of isometries, and give a brief idea of how to view complex hyperbolic space as a symmetric space.
References: [BH10][Chapter II.10, Sections: *The Group of Isometries of* $\mathbb{K}H^n$], [Par03][Section *3.5* for classification of isometries], [Gol99][Sections *3.1,3.2* for some more reference].

3. **The boundary and Heisenberg geometry (Max W.):** Recall the definition of the boundary at infinity, define horospheres, discuss horocyclic coordinates, define the Heisenberg geometry for the 2-dimensional complex hyperbolic space. If time permits, discuss the Heisenberg geometry for any dimension of complex hyperbolic space.
References: [BH10][Chapter II.10, *10.22, 10.23,10.29*], [Par03][Sections *4.1, 4.2* for 2-dimensional case] and [Gol99][Chapter *2.6* for general Heisenberg spaces, and chapter *4.2* for the Heiseberg geometry of the boundary].

4. **Totally geodesic submanifolds I (Carl):** Discuss the totally real totally geodesic submanifolds (also called Lagrangian submanifolds), show that they are isometric to the real hyperbolic space of lower dimension, discuss the $\mathbb{C}$-affine lines (also called complex lines or complex geodesics), show that they are isometric to a lower dimensional complex hyperbolic space. Sketch a proof that the only totally geodesic submanifolds are of this form, and deduce that there do not exist any totally geodesic hypersurfaces.
References: [BH10][Chapter II.10, Section *The Curvature of Distinguished Subspaces of* $\mathbb{K}H^n$], [Par03][Section *5.2, 5.3, 5.4* for the 2 dimensional case], and [Gol99][Sections *3.1.4* for complex lines, *3.1.9* for totally real subspaces, *3.1.11* for classification of totally geodesic submanifolds (use fact about Lie triple systems as a black box)].

5. **Totally geodesic submanifolds II, view from infinity (Marta):** Discuss chains, inversions, how any two points in the boundary can be joined by a chain [Gol99][4.3.5 (1)], foliations by orthogonal chains, and chain preserving transformations. Discuss $\mathbb{R}$-circles, how to represent them, and their moduli space.

References: [Gol99][Sections *4.3,4.4*] and [Par03][Section *5.5* for 2-dimensional case].

6. **Bisectors and spinal spheres (Fernando):** Introduce bisectors, spines, their decompositions, slices, the duality between bisectors and geodesics, automorphisms of bisectors, examples of bisectors, and elementary bisector intersections.
   References: [Gol99][Sections *5.1, 5.2, 5.3*].

7. **Kähler geometry (Arthur):** Review complex and Kähler geometry, symplectic geometry, and symplectic reduction. Then explain how to obtain the Kähler structure on complex hyperbolic space.
   References: [Gol99][Sections *2.1, 2.2, 2.3, 2.4* for review, and section *3.1.3* for the Kähler structure].

8. **Contact and CR geometry of the boundary (Tobias):** Define contact structures and CR geometry and then explain what these structures are on the boundary of complex hyperbolic space.
   References: [Gol99][Section *2.5* for preliminaries and section *4.2.4* for the CR structure on the boundary].

9. ~~***Arithmetic lattices:**~~ Discuss $\mathsf{SU}(n,1)$ as a (real) algebraic group, define lattices and arithmetic lattices, go over the construction of (cocompact) arithmetic lattices in $\mathsf{SU}(n,1)$. Mention (but no need to discuss!) the result in [BFMS20] about rigidity of arithmetic lattices.
   References: [Mor15][Sections *4.1, 5.1* and especially exercise *18.7 #1*].

10. ***Triangle groups I (Colin):** Introduce triangle groups, the angular invariant, and discuss the result in [GP92] about conditions on discreteness of triangle groups in complex hyperbolic space.
    References: [Gol99][Section *7.1* for the angular invariant], [GP92][For result on discreteness], and [Tho10][Section *2.1* for more on triangle groups].

11. * **Triangle groups II (Colin):** Discuss how to construct lattices in $\mathsf{PU}(2,1)$ using triangle groups, and mention how these can/are used to construct, in particular, non-arithmetic lattices.
    References: [DPP21][Discussion in section *1* and especially section *3* for triangle groups] and [Tho10][Section *2.1* for more on triangle groups].

References

[BFMS20] Uri Bader, David Fisher, Nicholas Miller, and Matthew Stover, *Arithmeticity, superrigidity and totally geodesic submanifolds of complex hyperbolic manifolds*, 2020, arXiv:2006.03008.

[BH10] Martin R. Bridson and André Haefliger, *Metric Spaces of Non-Positive Curvature*, Grundlehren der mathematischen Wissenschaften, Springer Berlin, Heidelberg, 2010.

[DPP21] Martin Deraux, John R. Parker, and Julien Paupert, *New Nonarithmetic Complex Hyperbolic Lattices II*, Michigan Mathematical Journal **70** (2021), no. 1, 133 – 205.

[Gol99] William M. Goldman, *Complex Hyperbolic Geometry*, Oxford Mathematical Monographs, Clarendon Press, 1999.

[GP92] William M. Goldman and John R. Parker, *Complex hyperbolic ideal triangle groups.*, Journal für die reine und angewandte Mathematik **425** (1992), 71–86.

[Mor15] Dave Witte Morris, *Introduction to Arithmetic Groups*, Deductive Press, 2015.

[Par03] John R. Parker, *Notes on Complex Hyperbolic Geometry*, 2003.

[Tho10] James M. Thompson, *Complex hyperbolic triangle groups*, Doctoral thesis, Durham University, 2010.