RTG Seminar: Group (Co)Homology

April 2022

The idea of this seminar is to get an understanding of the homology and cohomology theory of discrete groups.

Talks

(1) **Basic definitions** ....

The idea is to introduce the basic notions such as group rings, G-modules, injective and projective modules. Discuss briefly the so-called standard resolution or bar resolution and the uniqueness of resolutions.

References: [Bro12, Ch. I.0–3, Ch.I.5+.7] and [Lö, Ch. 1.1, Ch. 1.2.1].

(2) **Definition of $H_*(G;M)$ and $H^*(G;M)$** ....

We continue with some definitions: the G-invariants M^G and G-coinvariants M_G and the functors Tor and Ext. Then, define $H^*(G;M)$ and $H_*(G;M)$. Discuss examples: finite cyclic groups and free groups. It would be interesting to mention the case $M = \mathbb{Z}$.

References: [HS12, Ch. VI.2], [Wei96, Ch. 6.1], and [Lö, Ch. 1.6.3+.4, Ch. 3.1].

(3) **Classifying spaces for groups** ....

The goal is to get an intuition of what a $K(G,1)$ of BG space is. Define what a classifying space of G is, give an "homotopical characterization" and examples! (e.g. graphs, tori). Classifying space of a direct and free product. If time allows it maybe mention what do they classify and what a $K(G,n)$ is.

References: [Lö, Ch. 4.1.1–.2].

(4) **Group (co)homology via classifying spaces** ....

Explain the identification $H_*(G;\mathbb{Z}) \cong H_*(BG;\mathbb{Z})$ where the left hand side was defined in talk (2) and the right hand side is the singular homology. If time allows it define the local coefficient system $\mathcal{M}$ on BG associated to M and say a few words about the identification the $H_*(G;M) \cong H_*(BG;\mathcal{M})$. All this can be carried out also for $H^*(\cdot;\cdot)$.

References: [Lö, Ch. 4.1.4], [Bro12, Ch. III.1] and [Geo07, Ch. 8.1].

(5) **Group (co)homology in lower degrees** .... Daniel Roca (?)

Mention to which objects we have seen does H^0, H_0 relate. Explain what $H_1(G)$ is (you can save time using Hurewicz Theorem and what we saw in talk (4) or see [Wei96, Thm. 6.1.11]). State Hopf's Formula (no proof! this follows in talk (9)) for $H_2(G)$ and explain the classification of group extensions (with abelian kernel!) via $H^2(G)$.

References: [Lö, Ch. 1.5.2], [HS12, Ch IV.4, .9+.10], (and [Wei96, Thm. 6.1.11]).

(6) **Mayer-Vietoris sequence for amalgamated products** . . . .

Use the Mayer-Vietoris sequence to compute the homology of a group G by decomposing its classifying space BG. Consider free products ([Lö, Example 4.1.29])(compare the computations from talk (2)) and amalgamated products.

References: [Bro12, Ch. II.7 + Appendix] and [Lö, Example 4.1.29].

(7) **(Co)Homology of subgroups** . . . .

The idea is to understand how does the (co)homology of a subgroup relate to the (co)homology of the ambient group. Briefly explain Shapiro's lemma.

References: [Lö, Ch. 1.7], [Bro12, Ch. III.9+10], and [HS12, Ch. VI.16].

(8) **Basic on spectral sequences** . . . . Leonid Grau

The idea is to gain some intuition about spectral sequences and introduce the Hochschild-Serre spectral sequence.

References: [Lö, Ch. 3.2.1-.3].

(9) **Application of Hoschild-Serre spectral sequence** . . . .

Explain the 5-term exact sequence and give an outline of the proof of Hopf's formula seen on talk (5) with the techniques of talk (8). If time allows it maybe compute the group homology for the Dihedral group D_n.

References: [Lö, Ch. 3.2.4] and [HS12, Ch. IV.8].

(10) **Encore (aka Zugabe)** . . . .

Explain Künneth spectral sequence and mention how this gives us a version of the universal coefficient theorem and of Künneth's theorem.

References: [Lö, Ch. 3.2.4+.5].

References

[Bro12] Kenneth S Brown. *Cohomology of groups*, volume 87. Springer Science & Business Media, 2012.

[Geo07] Ross Geoghegan. *Topological methods in group theory*, volume 243. Springer Science & Business Media, 2007.

[HS12] Peter J Hilton and Urs Stammbach. *A course in homological algebra*, volume 4. Springer Science & Business Media, 2012.

[Lö] Clara Löh. Group cohomology.

[Wei96] Charles Weibel. An introduction to homological algebra. *Bulletin of the London Mathematical Society*, 28(132):322–323, 1996.