RTG Log Cabin Seminar: Boundaries

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The concept of geometric boundaries captures the asymptotic behavior of various geometric objects such as groups, spaces and manifolds. The set of boundary points offers ways to compactify our object of interest and above all, it often carries surprising geometry itself. However, the notion of a boundary is far from being canonical even for the same type of objects – there may be many inequivalent boundaries arising from different contexts.

The goal of this seminar is to introduce the different notions of boundaries in broad brushstrokes, with a focus on examples and applications. We will try to understand the motivation behind these definitions and discover how they evolve as we relax the rigidity of our objects. Furthermore, we will find out how boundaries can reveal more information about the objects we wish to study and see what keeps hidden.

List of Talks

Each talk is aimed to be 60-70 minutes to include time for questions and discussions. The first five talks are more elementary and set up some common ground, while the latter six talks branch into two topics: (a) the notion of boundary on groups and (b) the boundary of noncompact symmetric spaces. Depending on the number of participants willing to give talks, we may not be able to cover all of the topics.

1. **Non-positively curved spaces and groups**: This first talk aims to introduce the various different spaces and groups whose boundaries we wish to study. Introduce CAT(0)-*spaces*, δ-*hyperbolic spaces* and give two examples of each, including symmetric spaces and $\mathbb{H}^n$. Next, define the *Cayley graph* of a group, and give examples including that of the free group. Define *word-hyperbolic groups* using the Cayley graph and give two examples. Define *geometric group actions* and CAT(0)-*groups* and give two examples. Notice in each case their nonpositive curvature condition allows one to discuss geodesic rays going off to infinity. *State the Cartan-Hadamard Theorem.

   Suggested reference: [BH99] Part II.1, Part III.H.1, [Ebe97] 2.1, [Lue08] Sections 5-8.

2. **Visual boundary of complete CAT(0)-spaces**: Define the *visual boundary* ∂X of a complete CAT(0) space X and introduce the cone topology on $\overline{X} = X \cup \partial X$. Show that isometries of X naturally extend to homeomorphisms of $\overline{X}$, giving rise to a homeomorphism of the boundary. As examples, discuss $\partial \mathbb{H}^n$ and the topology of the visual boundary of one (or both) of the following: a regular tree or $\mathbb{R}$-tree.

Suggested reference: [BH99] Part II.8, [Ebe97] 1.7.

3. **Visual boundary as a metric space**: Introduce the *angular metric* $\angle$ on ∂X, show that it is a well-defined metric, and show that $(\partial X, \angle)$ is a complete metric space. Introduce the *Tits metric* on ∂X and state (not prove!) that it is a complete CAT(1) space for X a proper complete CAT(0) space. Discuss the relationship of the Tits boundary with the geometry of flats. *For examples, briefly sketch the boundary structure of $\mathbb{H}^2 \times \mathbb{H}^2$ or that of symmetric spaces. (Note: This talk comes after the discussion of symmetric spaces.)

   Suggested references: [BH99] Part II.9, [Ebe97] 3.1-3.4, *3.6.

4. **Gromov boundary of hyperbolic spaces**: Define *quasi-isometries* of metric spaces and demonstrate via an example that quasi-isometries may be much coarser than isometries. Introduce *quasi-geodesic rays* and define the *Gromov boundary* of a δ-hyperbolic space, and show, via the so-called Morse Lemma (Thm 1.7 [BH99]), that it is equivalent to the boundary defined using geodesic rays.

   Suggested reference: [BH99] Part I.8, Part III.H.1 Thm 1.7, Part III.H.3, [KB02] Sections 2.

5. **Gromov boundary and its properties**: State that the visual boundary for Gromov boundary for δ-hyperbolic spaces agrees with the visual boundary when the underlying space is complete CAT(0). Show that quasi-isometric embeddings induce a boundary map that is a homeomorphism when the embedding is a quasi-isometry, deducing that $\mathbb{H}^n$ is not quasi-isometric to $\mathbb{H}^m$ for $n \neq m$. Finally, show that the Gromov boundary is metrizable by constructing the *visual metric* on it.

   Suggested reference: [BH99] Part III.H.3.

Boundaries of Groups

6. **Boundary of hyperbolic groups:** Define the Gromov boundary of a hyperbolic group via its Cayley graph and show that it is a quasi-isometry invariant. Explain the Svarc-Milnor Lemma and as consequence the Gromov boundary of a δ-hyperbolic group is homeomorphic to that of a hyperbolic space on which there is a geometric group action (Thm 2.24 [KB02]). Use this to discuss the Gromov boundary of surface groups and free groups. Show that hyperbolic groups act on their Gromov boundaries as *uniform convergence groups*, the so-called "north-south" dynamics.

   Suggested Reference: [BH99] Part I.8 (especially 8,17), [KB02] Section 2, 4-5.

7. **Boundary of CAT(0)-groups:** In contrast to hyperbolic groups in the previous talk, show that the boundaries of CAT(0)-spaces are not well-defined for CAT(0)-groups, i.e. it is not a quasi-isometry invariant, counterexample provided by [CK00]. Introduce the *contracting boundary* of a CAT(0)-space, which a subset of the visual boundary [CS14], and show that it is a quasi-isometry invariant. This is a special case of a *Morse boundary* and provides a well-defined notion of boundary for CAT(0)-groups.

Briefly mention *sublinearly Morse boundaries* by giving an overview similar to the one in the introduction of [PQ23].

Suggested Reference: [CS14], [CK00], [PQ23].

8. ***Poisson and Martin boundaries:** You are free to decide to focus on one of the boundaries or discuss both. The following is a suggestion: Give an overview of the *Poisson and/or Martin boundaries* associated to *random walks* on a group or graph, introducing all the relevant definitions and compare the two notions. Discuss at least one interesting application, there are many in the references.

   Suggested reference: [KB02] Section 11, [Saw97], [KM98], [Giu] Week at Infinity, Random walks on hyperbolic groups.

Boundary of Symmetric Spaces

9. **Crash course in symmetric spaces**: Instead of going into the theory of symmetric spaces in full generality, focus on the example $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$, and introduce definitions and properties as they show up. This is a symmetric space of noncompact type, discuss its *Killing form*, *Cartan involution* and *Cartan decomposition* $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of the point $p = I \cdot \mathrm{SO}(n,\mathbb{R})$. Describe its *maximal abelian subspaces* of $\mathfrak{p}$ and discuss their relation to *maximal flats*. Finally, discuss its *root space decomposition*. (To simplify matters, you can take $n = 3$ throughout if you want).

   Suggested reference: [Ebe97] 2.1-2.10, 2.13.

10. **Furstenberg boundary and flag manifolds**: Continue the discussion of the example $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$. Describe *regular elements* of $\mathfrak{p}$, its *Weyl Chambers* and *Weyl group*, and explain how points at infinity are associated to an *eigenvalue-flag pair*. Each regular point at infinity determines a *parabolic subgroup* (classified in [Ebe97] 2.17.27) that agrees with any other such point in the same Weyl chamber. Introduce the *Furstenberg boundary* and show that it is well-defined. It is compact and comes equipped with a metric that naturally gives it a smooth manifold structure.

    Suggested reference: [Ebe97] 2.13, 2.17, 3.8.

11. ***Shilov boundary:** Discuss the *Shilov boundary* as a particular subset of the visual boundary of symmetric spaces and provide some examples or applications, e.g. the role they play in the context of maximal representations.

    Suggested Reference: Colin Davalo, [BIW10].

References

[BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature.* "https://egg.epfl.ch/metriques/Bridson-Haefliger.pdf". Springer, 1999.

[BIW10] Marc Burger, Alessandra Iozzi, and Anna Wienhard. "Surface group representations with maximal Toledo invariant". In: *Annals of Mathematics* 172.1 (2010), pp. 517–566. DOI: 10.4007/annals.2010.172.517.

[CS14] Ruth Charney and Harold Sultan. "Contracting boundaries of CAT(0) spaces". In: *Journal of Topology* 8.1 (2014), pp. 93–117.

[CK00] Christopher B. Croke and Bruce Kleiner. "Spaces with nonpositive curvature and their ideal boundaries". In: *Topology* 39.3 (2000), pp. 549–556.

[Ebe97] Patrick Eberlein. *Geometry of nonpositively curved manifolds.* University of Chicago Press, 1997.

[Giu] Tiozzo Giulio. *Random Walks on Weakly Hyperbolic Groups - Slides.* URL: http://www.math.toronto.edu/tiozzo/.

[KM98] Vadim A. Kaimanovich and Howard Masur. "The poisson boundary of Teichmüller space". In: *Journal of Functional Analysis* 156.2 (1998), pp. 301–332. DOI: 10.1006/jfan.1998.3252.

[KB02] Ilya Kapovich and Nadia Benakli. "Boundaries of hyperbolic groups". In: *Contemporary Mathematics* (2002). "https://faculty.math.illinois.edu/~kapovich/PAPERS/bry1.pdf", pp. 39–93.

[Lue08] Wolfgang Lueck. *Survey on geometric group theory.* 2008. URL: https://arxiv.org/abs/0806.3771.

[PQ23] Gabriel Pallier and Yulan Qing. *Sublinear biLipschitz equivalence and sublinearly Morse boundaries.* 2023. arXiv: 2211.01023 [math.GR].

[Saw97] Stanley A. Sawyer. "Martin boundaries and Random Walks". In: *Harmonic Functions on Trees and Buildings* (1997), pp. 17–44.