Morse Theory Seminar Plan

January 2024

Please talk to the people giving the talk before or after you on the same day if you don't have enough or too much material to cover.

1 Basics: Morse functions and Morse lemma (Alexander Mrinski)

Follow Chapter 2 of [4] and cover critical points, the Hessian of a function at a critical point, the Morse lemma and the index of a function at a critical point. Also define 1-parameter groups and state the fact that a vector field with compact support generates a 1-parameter group. Prove it if time permits. See also Chapter 1.1 and 1.3 in [1] as a second source.

2 Pseudo-Gradients, Morse-Charts and stable and unstable manifolds (David Lenze)

Explain Pseudo-Gradients, stable and unstable manifolds and prove that the trajectories of pseudo-gradients connect critical points. Cover sections 2.1a-d in [1].

3 Homotopy Type: The topology of sublevel sets (Lina Deschamps

Explain how critical points influence the structure of sublevel sets. Cover 2.1e-f in [1], this corresponds to (more general versions of) theorem 3.1 and theorem 3.2 in [4]. Don't be afraid to talk to the person before or after you to offload theorem 2.1.7 in [1].

4 Homotopy Type: CW structure and applications (Jannis Weis)

Explain how to use the topology of the sublevel sets to show that the manifold has a CW structure. Quickly define what a handle and a handle decomposition are (see [2]) and state that these Morse techniques can be used (with a bit more care) to prove the existence of handle decompositions for closed manifolds. Use what we have learned so far to prove the Reeb sphere theorem. Cover theorem

3.5 and theorem 4.1 in [4], see also 2.1.9 in [1]. Cover the CW structure on complex projective space as described in §4 of [4] if you have time.

5 Smale condition and Morse homology (Tom Stalljohann)

Introduce the Smale condition and state that generic pseudo-gradients satisfy the Smale condition. Prove that the index decreases along gradient lines (see section 2.2 in [1]). Define the Morse homology and explain the very rough idea of the proof (see section 3.1 in [1]). Also cover some of the basic examples where the Morse homology can be calculated, see again section 3.1 in [1].

6 Broken trajectories (Tobias Witt)

Introduce the space of broken trajectories. The goal of this talk is the result that the space of broken trajectories is a compact oriented 1-dimensional manifold with boundary. Define the topology on the space of broken trajectories. Sketch/give a proof that the space of broken trajectories is a compact 1-dimensional manifold with boundary. Use this result to conclude that the chain complex used to define Morse homology is actually a chain complex. The source for this talk is section 3.2 in [1]. It would be best if the person giving this talk is already familiar with the material.

7 $\mathbb{Z}$-coefficients and independence from Morse function and pseudo-gradient (Arthur Limonge)

Explain how we can use the results of the previous section to actually define a homology with $\mathbb{Z}$-coefficients, use section 3.3 in [1] as a source. Follow section 3.4 in [1] and show that the Morse homology is independent from the Morse function and the pseudo-gradient used to define it. State that the Morse homology is isomorphic to the singular homology. State how one can define the Morse homology of a manifold with boundary and note the connection to the relative homology, in the case where the boundary decomposes in a good way. See [1], sections 3.5, 4.1 and 4.9.

8 Handle decomposition of manifolds (João Lobo Fernandes)

Follow Chapters 1, 2 and 3 of [3]. Start by introducing (without proof) the notion of Morse functions for manifold triads (aka cobordisms) and explain that it recovers the relative homology groups of the manifold with respect to one end. Everything is analogous to what is set up for closed manifolds, so don't be afraid to skip the proofs as long as the statements are precise. Recall the definition of a handle attachment and a handle decomposition of a manifold triad. Introduce the notion of elementary cobordisms and their defining characteristic embeddings (Def. 3.9). Introduce the intersection number of two submanifolds and explain how the differential in the Morse complex can be described in terms of intersection numbers of the left and right spheres of handles.

9 The h-Cobordism Theorem and Generalized Poincaré Conjecture (João Lobo Fernandes)

Follow chapters 6,7 and 9 of [3]. Define h-cobodisms and state Theorem 9.1. Explain the main strategy, i.e. one wants to inductively cancel critical points. You can ignore cancellation of critical points of index 0 and 1. This is theorem 7.4. The main focus of this talk is Theorem 7.6 (aka the Basis Theorem). Sketch the proof, without worrying about Morse function technicalities, by mentioning that the heavy lifting is done by the Whitney trick (aka Theorem 6.6). If time permits, sketch the proof of Proposition B in chapter 9, i.e. the generalized Poincaré conjecture.

References

[1] M. Audin and M. Damian. *Morse theory and Floer homology.* Springer, 2014.

[2] Yukio Matsumoto. *An introduction to Morse theory*, volume 208 of *Translations of Mathematical Monographs.* American Mathematical Society, Providence, RI, 2002. Translated from the 1997 Japanese original by Kiki Hudson and Masahico Saito, Iwanami Series in Modern Mathematics.

[3] J. Milnor, L. Siebenmann, and J. Sondow. *Lectures on the H-Cobordism Theorem.* Princeton University Press, 1965.

[4] J. Milnor, M. Spivak, and R. Wells. *Morse Theory. (AM-51), Volume 51.* Princeton University Press, 1969.